

Nonsingular cosmology from evolutionary quantum gravity.

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We provide a cosmological implementation of the evolutionary quantum gravity, describing an isotropic Universe, in the presence of a negative cosmological constant and a massive (preinflationary) scalar field. We demonstrate that the considered Universe has a nonsingular quantum behavior, associated to a primordial bounce, whose ground state has a high occupation number. Furthermore, in such a vacuum state, the super-Hamiltonian eigenvalue is negative, corresponding to a positive emerging dust energy density. The regularization of the model is performed via a polymer quantum approach to the Universe scale factor and the proper classical limit is then recovered, in agreement with a preinflationary state of the Universe. Since the dust energy density is redshifted by the Universe deSitter phase and the cosmological constant does not enter the ground state eigenvalue, we get a late-time cosmology, compatible with the present observations, endowed with a turning point in the far future.

I. INTRODUCTION

The problem of defining a proper notion of time in quantum gravity is one of its most puzzling questions, common to the different existing approaches [1–3]. From a physical point of view, the relational approach [4] appears very promising, especially when matter is included in quantum dynamics. In fact, as discussed in [5–7] a dualism exists between a dust fluid and the time evolution of the quantum gravitational field.

However, such a promising approach has a weak point, concerning the nonpositive character of the super-Hamiltonian spectrum in comparison with the intrinsic positive energy density associated to a dust fluid.

The possibility to interpret the super-Hamiltonian eigenvalues as the comoving contribution $-\rho\sqrt{h}$ (ρ being the dust energy density and h the three-metric determinant) allows interesting speculations [8–10] on the role played in quantum gravity by a reference frame: no longer a simple gauge reparametrization of the dynamics, but a real source involved in the system evolution, *i.e.* a quantum violation of the general relativity principle [11]. Such a dualism between a dust fluid and a clock has a mandatory implementation in quantum cosmology, where the nonvanishing super-Hamiltonian spectrum must provide, on the classical limit, a dustlike cosmological component of the Universe, also investigated as a possible dark matter candidate [12, 13].

Here we provide a self-consistent picture of the quantum evolution of the early Universe, as described in the framework of evolutionary quantum gravity, showing how the resulting dust component has a positive energy density and its contribution to the Universe critical parameter is redshifted by the inflationary scenario to unobservable values. We consider a homogeneous and isotropic flat Universe, endowed with a negative cosmological constant and a massive scalar field, well mimicking the preinflationary behavior of the inflaton field [14, 15, 18] in a model with a decay from a false vacuum (see [16, 17] for the comparison of this scenario with experimental data).

We analyze the quantum evolution of this cosmological model according to a revised Wheeler-DeWitt approach which allows for a time evolution of the Universe wave function, as determined by a Schrödinger prescription. By an adiabatic approximation, we show how the considered quantum dynamics has a well-defined classical limit and at early stages predicts a nonsingular behavior (removing the initial singularity), in close analogy to a big-bounce structure in the limit of very small Universe volumes. Furthermore, we

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calculate the full spectrum of the super-Hamiltonian, which is associated to a positive value of the dual dust energy density, but with the shortcoming of an unbounded-from-below profile of the eigenvalues.

To remove the unpleasant feature of an unstable quantum system which does not possess a ground state, we treat the Universe volume as a discrete variable in the polymer quantum approach [19–21]. As a result, the positive nature of the spectrum is altered (states corresponding to negative dust energy density appear), but a stable ground state emerges in correspondence to very high occupation numbers. Such a ground (vacuum) state is associated to a positive dust energy density and it is suitable to implement a quasiclassical limit of the dynamics. In fact, in the limit $\hbar \rightarrow 0$, the high occupation number of this ground state ensures finite energy density, which is of Planckian value if the discretization parameter of the polymer approach is taken of the Planck length order. It is worth stressing how the value of the negative cosmological constant does not enter the ground state eigenvalue and it can be taken sufficiently small to ensure a turning point of the Universe in the far future (de facto the presented model describes a cyclic Universe, possessing a future classical turning point and a bounce in the past).

In the proposed scheme, the preinflationary Universe emerges as the classical limit of an evolutionary quantum dynamics and it is endowed by a dust energy density (relic of the quantum clock) which is then redshifted to very small values by the subsequent de Sitter phase. We stress how the dust contribution, differently from the ordinary matter, cannot be restored by the inflaton decay during the reheating phase and its fate is an increasing dilution up to present unobservable amounts. In other words, the impossibility to observe today the matter counterpart of the evolutionary quantum gravity model is explained in the same spirit as the “unwanted relic paradox” is solved by inflation [14, 15].

Therefore, we provide a very promising cosmological implementation of the evolutionary quantum gravity, thought as the intrinsic matter-time dualism, which relies on the existence of a regular ground state of the Universe, endowed with appropriate properties for a regular classical limit.

In summary, while the problem of introducing a reliable time variable in quantum gravity is clearly an open question, and many competitive approaches stand in literature [2, 3, 22, 23], we emphasize how the present analysis enforces the idea that the time-fluid dualism admits intriguing cosmological applications, as already inferred in [13]. In fact, the most significant difficulty of such a relational approach is the nonpositive nature of the super-Hamiltonian spectrum, resulting in the nonpositive energy density of the fluid. We propose as solution the restriction of such a requirement to the ground state of the theory only. Here, we achieve a positive regularized “vacuum” state for the model, via the introduction of a negative cosmological constant, but its validity can be more general. In this sense, the matter-time dualism possesses a nice feature in the cosmological paradigm, which upgrades its reliability, making it of comparable impact to the well-known multitime approach [27], which is very powerful for characterizing the cosmological evolution in terms of the Universe volume as time variable.

The present paper is organized as follows: in Sec. II we present the framework of evolutionary quantum gravity in a minisuperspace context; in Sec. III we give some physical arguments on why we consider an evolutionary quantum cosmology and a negative cosmological constant; in Sec. IV we introduce the cosmological model and we solve the associated evolutionary Schrödinger equation in the early- and late-time limits; in Sec. V we perform polymer quantization as $\rho \rightarrow 0$ and we outline the emergence of a bounded-from-below energy spectrum. Finally, in Sec. VI brief conclusions follow.

II. EVOLUTIONARY QUANTUM COSMOLOGY

One of the most promising approaches to the problem of time in quantum gravity relies on the dualism existing between a dust fluid and a physical clock [5–7]. Here we briefly discuss this correspondence in the framework of the minisuperspace, *i.e.* as restricted to the case of homogeneous cosmological models.

The cosmological implementation of the Wheeler-DeWitt equation corresponds to dealing with a finite number of degrees of freedom, say q_i ($i = 1, 2, \dots, n$) generalized coordinates, representing scale factors of the Universe and matter fields. The dynamics of the model is summarized by the classical system Hamiltonian $H(q_i, p_i)$, p_i being the conjugate momenta to the generalized coordinates. Implementing an evolutionary quantum dynamics for the considered homogeneous model, in place of the standard Wheeler-DeWitt frozen formalism [28], consists of assuming that the Universe wave function ψ evolves with respect to an external parameter t , which plays the role of a physical clock; *i.e.* we take $\psi = \psi(t, q_i)$.

The evolution of the system is then naturally determined by the Schrödinger equation

$$i\hbar\partial_t\psi = N(t)\hat{\mathcal{H}}\psi, \quad (1)$$

where $N(t)$ denotes the lapse function and $\hat{\mathcal{H}}$ is the operator version of the super-Hamiltonian. By taking

the wave function in the following integral representation

$$\psi(t, q_i) = \int dE \phi(E, q_i) \exp \left\{ -\frac{i}{\hbar} E \int N(t) dt \right\}, \quad (2)$$

the Schrödinger equation above is associated to the time independent eigenvalue problem

$$\hat{\mathcal{H}}\phi = E\phi, \quad (3)$$

E being the super-Hamiltonian eigenvalue.

As far as we take the classical limit, for $\hbar \rightarrow 0$, by setting the wave function as $\psi \sim \exp i\sigma/\hbar$, the eigenvalue problem above takes the form of an Hamilton-Jacobi equation, containing an additional matter contribution $-E$, σ being the Jacobi function. Since the Hamiltonian is a scalar density of weight $1/2$, to get an energy density, we need to divide this new contribution by the Universe volume $V_u \sim \sqrt{h}$ (h being the three-metric determinant) and hence we get $\rho_{new} = -E/V_u$, which clearly describes a dust comoving fluid.

The clock-dust-fluid dualism is well expressed in quantum cosmology by the correspondence traced above between the super-Hamiltonian spectrum and the energy density of the dust emerging in the classical limit of the evolutionary quantum picture. The limit of this analogy is in the nonpositive nature of the super-Hamiltonian spectrum, which prevents us from ensuring an always positive dust energy density. Indeed, we stress how, invoking a minimal energy principle for the quantum Universe, the request of a positive dust fluid energy density must be transferred to such a ground (vacuum) state only. In this respect, we finally observe how the value of the emerging contribution ρ_{new} depends directly on the boundary conditions, characterizing the considered cosmological model, which fixes the ground state eigenvalue. As a result, it is not immediately recognizable that the constraint ensuring that the emerging dust behaves as a test fluid. In this sense, the emerging energy density must be regarded as the physical substantiation of the comoving reference frame. In other words, an evolutionary quantum cosmology predicts, in the classical limit, the existence of a natural preferred comoving reference frame, unavoidably affecting the quantum evolution of the system. Despite the fact that the coordinate reparametrization of the adopted scheme is still allowed, the present scenario can be interpreted as quantum breaking of the general relativity principle [10].

III. PHYSICAL GROUNDS AND MOTIVATION

We now address two subtle features at the ground of our model, which are indeed intrinsically connected to each other, *i.e.* the implementation of an evolutionary quantum cosmology and the presence of a negative cosmological constant in the Universe dynamics.

Clearly, the solution of the problem of time is one of the open questions in quantum gravity, both for what concerns the nature of the field, materializing the clock, and also for the properties time must possess on a classical regime, as well as on a quantum regime. This problem was interestingly addressed in [1, 4, 24], where the relation between defining a time and defining a reference frame in quantum gravity are related concepts. For some attempts to characterize the time variable as a relational clock having peculiar properties, like a monotonic behavior, see [25, 26]. However, one of the most interesting points of view on this topic, different from the one here proposed, is the so-called multitime approach [27], in which it is emphasized how the gravitational field has to be separated into its two real physical degrees of freedom, while the remaining part of the space geometry labels the evolution. For a review of the various viable approaches to the problem of time in quantum gravity and a discussion of their successes and shortcomings, see [2] (see also [3]).

However, as discussed in the previous section, the dualism between an external time and the presence of a dust fluid in the quantum dynamics is a well-established fact. The physical nature of time depends on the details of the considered model and it relies on the link existing between the super-Hamiltonian eigenvalue and the dust energy density [5–7]. In the present analysis we avoid the specification of a particular framework, making essentially reference to the Schrödinger-like dynamics of the Universe wave function, because on a cosmological setting the physical output of an evolutionary approach is just the emergence of a new matter contribution to the thermal bath of the Universe. Such an additional matter density defines also a new reference frame, which is, to some extent, a preferred one, in view of its non-test-fluid character.

Hence, it is clear how the question concerning the positive value of the emerging dust energy density (a central theme in the matter-time dualism in quantum gravity) is crucial when reconstructing the Universe thermal history [14]. It is just the request to deal with a fully negative spectrum of the super-Hamiltonian (fully positive dust energy density) which leads us to involve a negative cosmological constant

in the dynamics. In so doing we get a contribution in the Hamiltonian which corresponds to a harmonic oscillator energy, but with a global negative sign. Such a feature has a rather general validity (see Sec. VI) since, even for a generic inhomogeneous cosmological model the kinetic contribution of the Universe volume to the Hamiltonian, together with the negative cosmological constant term, provides the same harmonic-oscillator structure. However, such a contribution implies an unstable behavior of the full Hamiltonian operator, whose spectrum is unbounded from below. The regularization is performed via a polymer quantum treatment of the Universe scale factor (say the Universe volume). Such a regularization procedure provides a well-defined ground state, having the right negative eigenvalue for getting a viable phenomenology. In other words, as proposed in [13], we are implementing the idea that, on a quantum level, it is enough to require that the dust energy density corresponding to the ground state be positive in order to get a reliable physical model.

It is worth stressing how the ground state eigenvalue is surprisingly unaffected by the value of the negative cosmological constant [see (64)] and, therefore, Λ is freely available for the cosmological problem and it can be properly fixed to a very small value, whose dynamical role will be only to provide a turning point in the far future. The negative cosmological term, here considered, has no “interference” with the present phenomenon of Universe acceleration [29, 30], whose origin could be due to a positive cosmological constant [31], but also to other physical mechanisms (as for instance quintessence paradigms [32]). This effect is indeed a late evolutionary feature of the Universe, almost uncorrelated with the domain of validity of this study. Furthermore, a positive cosmological constant term would not guarantee the spectral features traced above.

However, motivations for including a negative cosmological constant in the Universe dynamics can be found in many modern fundamental theories, such as supersymmetry [33] and AdS/CFT correspondence [34].

Finally, we stress how, independently of the dust energy density origin (a nonvanishing eigenvalue for the super-Hamiltonian or a matter clock involved in the dynamics) the corresponding cosmological evolution could have a deep phenomenological impact on the present Universe. We demonstrate that this contribution, associated to the regularized ground state eigenvalue, admits a reliable classical limit and then it is redshifted by the inflation e -folding, leaving no measurable trace on the observed Universe.

IV. THE MODEL

Let us consider a preinflationary Universe described by a flat Friedmann-Robertson-Walker (FRW) metric, whose metric is (we work in units $c = 1$)

$$ds^2 = dt^2 - a^2(t)dl_{RW}^2, \quad (4)$$

with

$$dl_{RW}^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (5)$$

and a massive noninteracting scalar field ϕ , modeling the inflaton trapped into a false vacuum.

The Hamiltonian for such a system in the presence of a negative cosmological constant $-\Lambda$, $\Lambda > 0$, reads [43]

$$\mathcal{H} = -\frac{2\pi G}{3}\frac{p_a^2}{a} - \frac{\Lambda}{8\pi G}a^3 + \frac{1}{2\hbar a^3}p_\phi^2 + \frac{1}{2}\frac{m^2}{\hbar}a^3\phi^2, \quad (6)$$

p_a and p_ϕ being the conjugate momenta to a and ϕ , respectively, while m is the mass of the scalar field.

In view of quantization a convenient set of phase-space coordinates is obtained via the canonical transformation

$$a \rightarrow \rho = a^{3/2}, \quad p_a \rightarrow p_\rho = \frac{2}{3}p_a\rho^{-1/3}, \quad (7)$$

and the Hamiltonian (6) in the new set of variables takes the following form

$$\mathcal{H} = -\frac{3\pi G}{2}p_\rho^2 - \frac{\Lambda}{8\pi G}\rho^2 + \frac{1}{2\hbar\rho^2}p_\phi^2 + \frac{1}{2}\frac{m^2}{\hbar}\rho^2\phi^2. \quad (8)$$

The canonical quantization of the associated dynamical system is obtained by replacing the configuration variables ρ and ϕ with multiplicative operators defined in a suitable (pre-)Hilbert space and the momenta with the proper derivative operators, *i.e.*

$$p_\rho \rightarrow -i\hbar\frac{\partial}{\partial\rho}, \quad p_\phi \rightarrow -i\hbar\frac{\partial}{\partial\phi}, \quad (9)$$

such that the Wheeler-DeWitt operator $\hat{\mathcal{H}}$ becomes

$$\hat{\mathcal{H}}\Psi(\rho, \phi) = \left[\frac{3\pi\ell_P^4}{2G} \frac{\partial^2}{\partial \rho^2} - \frac{\Lambda}{8\pi G} \rho^2 - \frac{\hbar}{2\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{2} \frac{m^2}{\hbar} \rho^2 \phi^2 \right] \Psi(\rho, \phi), \quad (10)$$

$\ell_P = \sqrt{\hbar G}$ being the Planck length. The evolutionary quantum equation associated with (10) reduces to an eigenvalue equation for the Hamiltonian (8)

$$\hat{\mathcal{H}}\Psi(\rho, \phi) = E\Psi(\rho, \phi). \quad (11)$$

A. Quasiclassical limit

Let us address a Born-Oppenheimer approximation (to be verified *a posteriori*), in which we construct the wave function as the product of the following two terms

$$\Psi(\rho, \phi) = \zeta(\rho) \chi(\rho, \phi), \quad (12)$$

in which χ describes the scalar field wave function and depends parametrically on ρ , while ζ is the Universe wave function.

Hence, we impose

$$\left[-\frac{\partial^2}{\partial \rho^2} + \frac{\Lambda}{12\pi^2 \ell_P^4} \rho^2 \right] \zeta_n(\rho) = \frac{2G}{3\pi \ell_P^4} E_n^\rho \zeta_n(\rho). \quad (13)$$

The equation above can be written as the Schrödinger equation describing a harmonic oscillator with mass $M = \frac{1}{3\pi G}$ and frequency $\Omega = \frac{1}{2} \sqrt{3\Lambda}$. A solution of the eigenvalue problem (13) is thus given by

$$\zeta_n(\rho) = N_n \Lambda_\star^{1/8} e^{-\frac{\sqrt{\Lambda_\star} \rho^2}{2}} H_n \left(\Lambda_\star^{1/4} \rho \right), \quad (14)$$

where $H_n(x)$ is the Hermite polynomial of degree n , $N_n = \sqrt{\frac{1}{\pi^{1/2} 2^n n!}}$ is a normalization constant and

$$\Lambda_\star \equiv \frac{\Lambda}{12\pi^2 \ell_P^4}. \quad (15)$$

The eigenvalues E_n^ρ reads

$$E_n^\rho = \hbar \frac{1}{2} \sqrt{3\Lambda} \left(n + \frac{1}{2} \right), \quad (16)$$

where $n \in \mathbb{N}$ is restricted to be odd [44].

In the same way, we impose

$$\left[-\frac{\partial^2}{\partial \phi^2} + \frac{m^2}{\hbar^2} \rho^4 \phi^2 \right] \chi_k(\rho, \phi) = \frac{2\rho^2}{\hbar} E_k^\phi \chi_k(\rho, \phi), \quad (17)$$

whose solution reads

$$\chi_k(\rho, \phi) = N_k m_\star^{1/4} \sqrt{\rho} e^{-\frac{m_\star \phi^2 \rho^2}{2}} H_k(\sqrt{m_\star} \phi \rho), \quad (18)$$

where

$$m_\star \equiv \frac{m}{\hbar}, \quad (19)$$

with eigenvalues given by

$$E_k^\phi = m \left(k + \frac{1}{2} \right), \quad k \in \mathbb{N}. \quad (20)$$

The solution of (8) is to be considered in the context of a theory of small oscillations of the scalar field around the minimum of its potential. In this way the scalar field cannot explore the complete profile of

the potential $V(\phi)$, and this makes the quantum number k confined in a limited interval whose upper limit depends on the parameters of the model.

In the appendix we demonstrate that an approximated solution of the eigenvalue equation (11) is given by the product of the functions (14) and (18) for $\rho \rightarrow \infty$, *i.e.*

$$\Psi_{n,k}(\rho, \phi) = \zeta_n(\rho) \chi_k(\rho, \phi) = N_n N_k \Lambda_\star^{1/8} m_\star^{1/4} e^{-\frac{\sqrt{\Lambda_\star} \rho^2}{2}} H_n(\Lambda_\star^{1/4} \rho) \sqrt{\rho} e^{-\frac{m_\star \phi^2 \rho^2}{2}} H_k(\sqrt{m_\star} \phi \rho), \quad (21)$$

with associated eigenvalue

$$\begin{aligned} E_{n,k} &= -E_n^\rho + E_\rho^\phi + \frac{\sqrt{3}}{2} \hbar \sqrt{\Lambda} \left(k + \frac{1}{2} \right) \\ &= \frac{\sqrt{3}}{2} \hbar \sqrt{\Lambda} (k - n) + m \left(k + \frac{1}{2} \right). \end{aligned} \quad (22)$$

Therefore, in the late-time limit $\rho \rightarrow \infty$, the Born-Oppenheimer approximation (21) is well grounded for energy eigenstates. However, the energy spectrum is unbounded from below; thus the resulting dynamical system is unstable.

Semiclassical states can be constructed through the wave packets peaked around some classical values \tilde{n} and \tilde{k} as follows

$$\Psi^{[\tilde{n}, \tilde{k}]}(\rho, \phi, t) = A \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} e^{-\frac{(n-\tilde{n})^2}{2\sigma_1^2}} e^{-\frac{(k-\tilde{k})^2}{2\sigma_2^2}} \zeta_n(\rho) \chi_k(\rho, \phi) e^{-i\frac{E_{n,k}t}{\hbar}}, \quad (23)$$

where σ_1 and σ_2 denote the distribution variances, while A is a normalizing factor.

We want to study the evolution in time of the expectation values of the operator ρ on such states, *i.e.*

$$\langle \rho \rangle_t = \left\langle \Psi^{[\tilde{n}, \tilde{k}]}(t) | \rho | \Psi^{[\tilde{n}, \tilde{k}]}(t) \right\rangle = \int_0^\infty d\rho \int_{-\infty}^\infty d\phi \left(\Psi^{[\tilde{n}, \tilde{k}]}(\rho, \phi, t) \right)^* \rho \Psi^{[\tilde{n}, \tilde{k}]}(\rho, \phi, t), \quad (24)$$

and its variation

$$\langle \Delta \rho^2 \rangle_t = \int_0^\infty d\rho \int_{-\infty}^\infty d\phi \left[\left(\Psi^{[\tilde{n}, \tilde{k}]}(\rho, \phi, t) \right)^* \rho^2 \Psi^{[\tilde{n}, \tilde{k}]}(\rho, \phi, t) \right] - \langle \rho \rangle_t^2. \quad (25)$$

A similar analysis is performed for ϕ .

In Figs. 1, 2, 3 and 4, the behaviors of such expectation values and variances are sketched.

We fixed $\tilde{n} = 11$, $\tilde{k} = 3$, $\sigma_1 = 0.1$ and $\sigma_2 = 0.1$, while $\Lambda_\star = 1$ and $m_\star = 100$. Time is in units of $\frac{2\pi}{10} \frac{1}{\Omega} = \frac{2\pi}{5\sqrt{3}\Lambda}$.

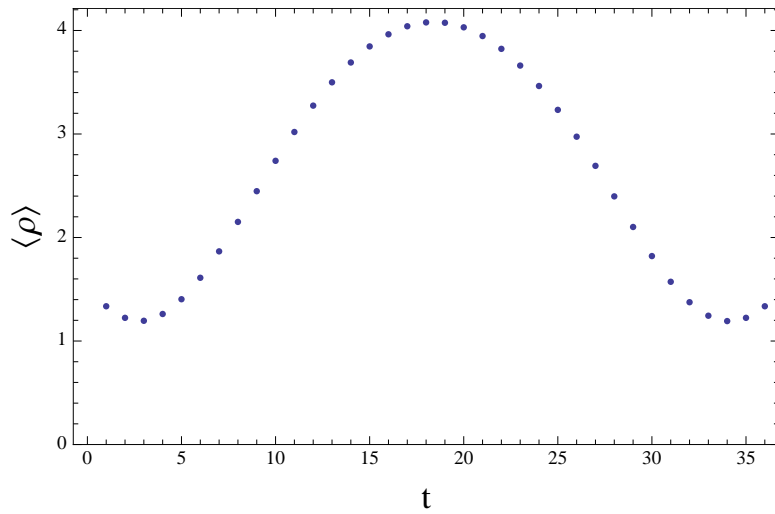


Figure 1: The points denote the expectation value of the scale factor on the wave packet (23) in units of $\frac{1}{\Lambda_\star^{1/4}}$.

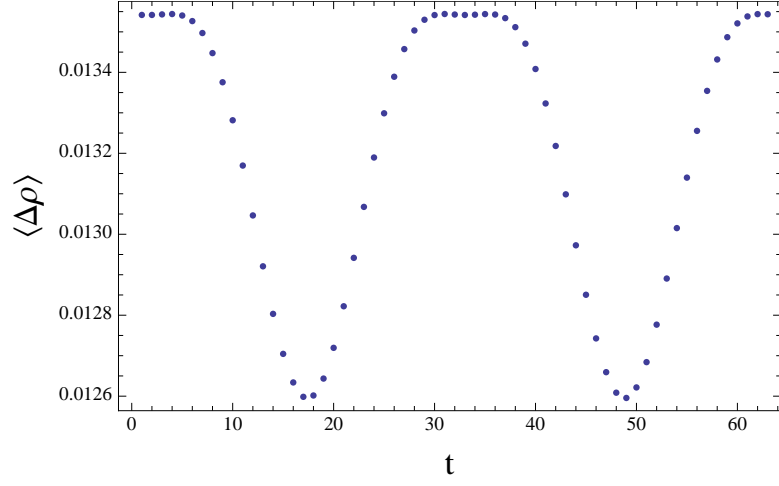


Figure 2: The uncertainty $\langle \Delta \rho \rangle_t$ calculated on the wave packet (23). $\langle \Delta \rho \rangle_t$ is in units of $\frac{1}{\Lambda_\star^{1/4}}$.

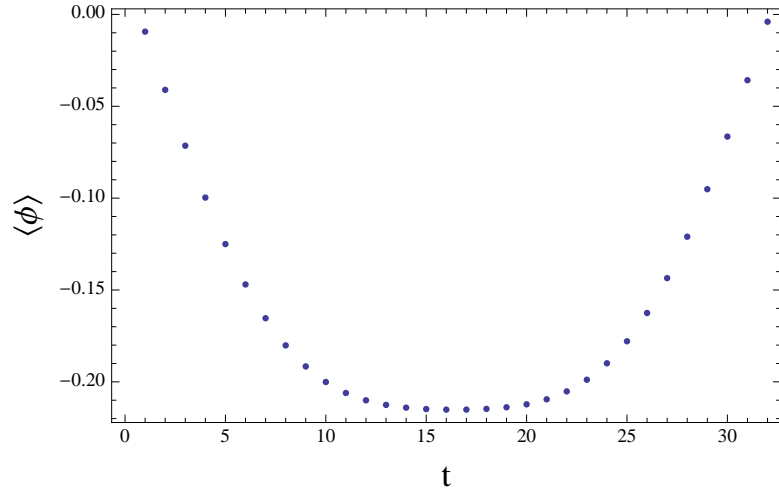


Figure 3: The expectation value of the scalar field calculated on the wave packet (23). $\langle \phi \rangle_t$ is in units of $\frac{\Lambda_\star^{1/4}}{\sqrt{m_\star}}$.

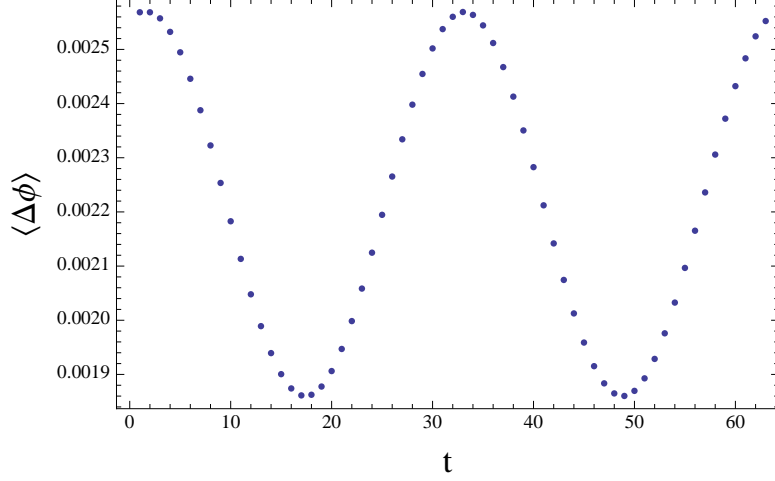


Figure 4: Motion of the scalar field uncertainty $\langle \Delta \phi \rangle_t$ calculated on the wave packet (23). $\langle \Delta \phi \rangle_t$ is in units of $\frac{\Lambda_*^{1/4}}{\sqrt{m_*}}$.

We see how the variances remain bounded and much smaller than the corresponding expectation values.

B. Close to the singularity

In the limit $\rho \rightarrow 0$ we expect the potential term of the scalar field in (8) to be negligible with respect to the kinetic one, *i.e.*

$$\frac{1}{2\rho^2}p_\phi^2 \gg \frac{1}{2}m^2\rho^2\phi^2, \quad (26)$$

such that the Hamiltonian (8) can be rewritten by neglecting this potential term

$$\mathcal{H} = -\frac{3\pi G}{2}p_\rho^2 - \frac{\Lambda}{8\pi G}\rho^2 + \frac{1}{2\hbar\rho^2}p_\phi^2. \quad (27)$$

Let us first consider the associated classical system coming out of an evolutionary quantum dynamics, *i.e.* for $\mathcal{H} = E$. By recalling the expression $p_a = -\frac{3c^2}{4\pi G}a\dot{a}$, the following Friedmann equation is obtained

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left(\frac{1}{2\hbar a^6}p_\phi^2 - \frac{E}{a^3} - \frac{\Lambda}{8\pi G} \right) \quad (28)$$

where p_ϕ is constant since the field is sufficiently frozen near the minimum. An analytic solution is given by

$$a(t) = [A + B \sin(\omega t + \phi)]^{1/3}, \quad (29)$$

where

$$A = -4\pi G \frac{E}{\Lambda}, \quad B = \pm \sqrt{\frac{8\pi G}{\Lambda} \left(\frac{p_\phi^2}{2\hbar} + 2\pi G \frac{E^2}{\Lambda} \right)}, \quad \omega = \sqrt{3\Lambda}, \quad (30)$$

and since $|B| > |A|$ the classical initial singularity $a = 0$ is still present. Indeed, it can be shown that there is also a final *big crunch* singularity, as usual in models with a negative cosmological constant.

The scalar field dynamics can also be solved analytically, finding

$$\phi = \frac{p_\phi}{\hbar\omega} \frac{2}{\sqrt{B^2 - A^2}} \operatorname{arctanh} \left(\frac{B}{\sqrt{B^2 - A^2}} + \frac{A}{\sqrt{B^2 - A^2}} \tan \left(\frac{1}{2}(\omega t + \phi) \right) \right) + \text{const.} \quad (31)$$

The classical behavior of a and ϕ is depicted in Figs. 5 and 6.

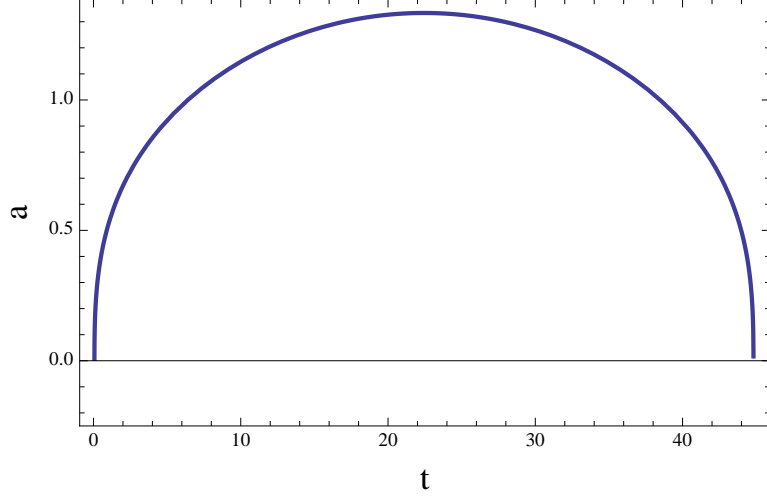


Figure 5: Classical trajectory of the scale factor a in function of time. Time is in units of $\frac{2\pi}{10\omega}$ and the scale factor $a(t)$ in units of $B^{1/3}$. It is worth noting how the singularity $a = 0$ is reached in a finite amount of time both in the past and in the future.

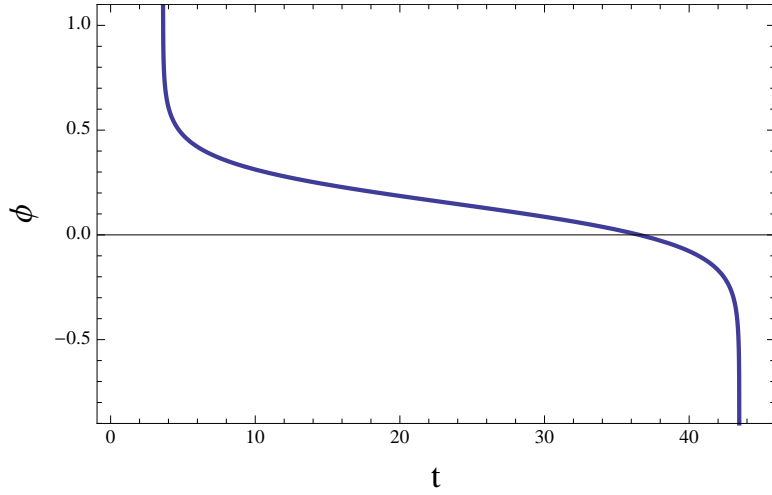


Figure 6: Classical trajectory of ϕ in function of time. Time is in units of $\frac{2\pi}{10\omega}$, the scalar field in units of $\frac{p_\phi}{\hbar\omega} \frac{2}{\sqrt{B^2 - A^2}}$.

It is worth noting that the presence of a negative cosmological constant ensures the existence of a turning point in the Universe late classical evolution.

The associated quantum equation in evolutionary quantum gravity reads

$$\left[\frac{3\pi\ell_P^4}{2G} \frac{\partial^2}{\partial \rho^2} - \frac{\Lambda}{8\pi G} \rho^2 - \frac{\hbar}{2\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \Psi(\rho, \phi) = E \Psi(\rho, \phi), \quad (32)$$

We look for a plane-wave solution for the scalar field, *i.e.*

$$\Psi(\rho, \phi) = \frac{1}{\sqrt{2\pi}} e^{i \frac{k_\phi \phi}{\hbar}} \zeta(\rho), \quad (33)$$

such that one gets the following equation for ζ

$$\left[\frac{\partial^2}{\partial \rho^2} - \Lambda_\star \rho^2 + \frac{k_{\phi\star}^2}{\rho^2} \right] \zeta(\rho) = E_\star \zeta(\rho). \quad (34)$$

with

$$k_{\phi\star}^2 = \frac{G}{3\pi\hbar\ell_P^4} k_\phi^2, \quad E_\star = \frac{2G}{3\pi\ell_P^4} E. \quad (35)$$

A solution of Eq.(34) can be constructed by considering the following wave function

$$\zeta = f(\rho) e^{\frac{-\sqrt{\Lambda_\star}\rho^2}{2}} \rho^\lambda, \quad (36)$$

where [45]

$$\lambda = \frac{1 + \sqrt{1 - 4k_{\phi\star}^2}}{2}. \quad (37)$$

In what follows, we consider the solution with real λ 's, *i.e.*

$$|k_{\phi\star}| < \frac{1}{2}. \quad (38)$$

We assume a power series expansion for the unknown function $f(\rho)$

$$f(\rho) = \sum_{n=0}^{n'} c_{n,n'} \rho^n, \quad n, n' \in 2\mathbb{N}, \quad (39)$$

and the eigenvalue problem (36) provides the following difference equation

$$c_{n+2,n'}(n+2) \left(\left(\sqrt{1 - 4k_{\phi\star}^2} + n + 2 \right) \right) - c_{n,n'} \left(E_\star + \sqrt{\Lambda_\star} \left(\sqrt{1 - 4k_{\phi\star}^2} + 2n + 2 \right) \right) = 0, \quad (40)$$

with the restriction

$$E_\star = -\sqrt{\Lambda_\star} \left(\sqrt{1 - 4k_{\phi\star}^2} + 2n' + 2 \right). \quad (41)$$

The expression of the coefficients $c_{n,n'}$ can be written in terms of (the analytically continued) Euler Γ -function as follows [35]

$$c_{n,n'}^{sol} = \frac{((-1)^n + 1) \Gamma \left(1 + \frac{1}{2} \sqrt{1 - 4k_{\phi\star}^2} \right) \Lambda_\star^{n/4}}{\Gamma \left(\frac{n}{2} + 1 \right) \Gamma \left(\frac{n}{2} + \frac{1}{2} \sqrt{1 - 4k_{\phi\star}^2} + 1 \right)} (-1)^{\frac{n}{2}} \frac{\frac{n'}{2}!}{\left(\frac{n'}{2} - \frac{n}{2} \right)!}. \quad (42)$$

and the solution of Eq. (32) thus reads

$$\Psi_{n',k_\phi}(\rho, \phi) = C e^{i \frac{k_\phi \phi}{\hbar}} e^{\frac{-\sqrt{\Lambda_\star}\rho^2}{2}} \rho^{\frac{1 + \sqrt{1 - 4k_{\phi\star}^2}}{2}} \sum_{n=0}^{n'} c_{n,n'}^{sol} \rho^n. \quad (43)$$

where C is a normalizing factor and with energy eigenvalue

$$E_{n',k_\phi} = -\hbar \frac{\sqrt{3\Lambda}}{4} \left(\sqrt{1 - 4k_{\phi\star}^2} + 2n' + 2 \right). \quad (44)$$

Indeed, the procedure we used to solve Eq. (34) is just the generalization of those adopted for a three-dimensional harmonic oscillator, in which case $k_{\phi\star}^2 = -l(l+1)$, l being the angular momentum. In fact, the eigenfunction (43) and the eigenvalue (44) correspond to the three-dimensional generalization of the harmonic oscillator eigenfunctions and eigenvalues. In particular, the sum $\sum_{n=0}^{n'} c_{n,n'}^{sol} \rho^n$ coincides with Laguerre polynomials generalized for continuous k_ϕ values. Within this model, the value of $k_{\phi\star}$ determines if there is a physical singularity [46] in $\rho = 0$: it is absent for $k_{\phi\star} < \frac{1}{2}$ and present for $k_{\phi\star} > \frac{1}{2}$ [36, 37]. Therefore, by assuming the condition (38), we get a nonsingular scenario for the early Universe and the big-bang singularity is removed.

Semiclassical states can now be constructed as follows

$$\Psi^{[\tilde{k}_\phi, \tilde{n}]}(\rho, \phi, t) = A \int_{-1/2}^{+1/2} dk_{\phi\star} e^{-\frac{(k_\phi - \tilde{k}_\phi)^2}{2\alpha^2}} \sum_{n=1}^{\infty} e^{-\frac{(n - \tilde{n})^2}{2\sigma^2}} e^{-i \frac{E_{n,k_\phi} t}{\hbar}} \Psi_{n,k_\phi}(\rho, \phi). \quad (45)$$

A being a normalizing factor. The behavior of expectation values and distribution variances is presented in Figs. 7 and 8 for $\sigma = 0.1$.

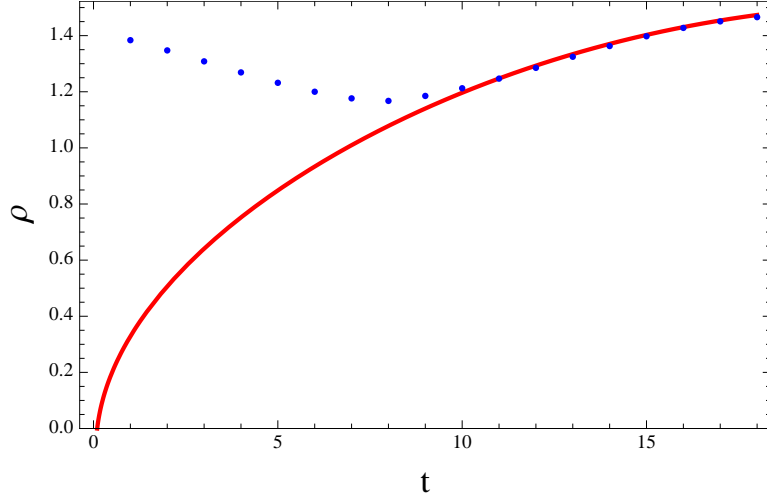


Figure 7: The points denote the expectation value of the scale factor on the wave packet (45), while the continuous line is the classical behavior of ρ (both in units of $\frac{1}{\Lambda_\star^{1/4}}$). We note how the initial singularity is avoided in the quantum model.

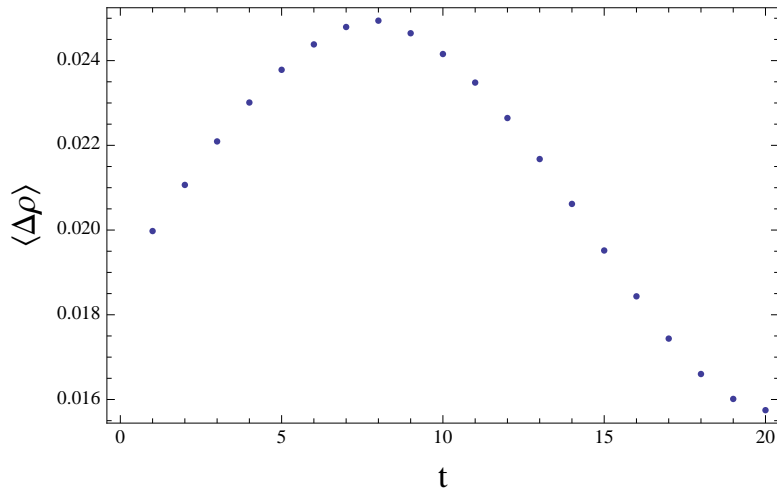


Figure 8: The uncertainty of ρ on the wave packet (45). $\langle \Delta \rho \rangle_t$ is in units of $\frac{1}{\Lambda_\star^{1/4}}$.

This analysis confirms that the quantum model is nonsingular as $\rho \rightarrow 0$, since the initial singularity is replaced by a bounce. Furthermore, deviation becomes smaller and smaller as t increases; thus the Universe becomes more and more classical at late times.

V. POLYMER QUANTIZATION

In this section we implement polymer quantization for the variable ρ describing the Universe volume. We will outline how this procedure provides a bounded-from-below dust energy contribution, which allows us to fix the value of E owing to the relaxation of the Universe to the fundamental state.

Polymer quantization [19, 20] realizes a representation of the Weyl algebra which is not unitary equivalent to the Schrödinger representation. In fact, it is based on violating one of the hypotheses of the Stone-von Neumann uniqueness theorem, namely weak-continuity. This is due to the fact that the space of configuration variables is endowed with a discrete topology, such that only finite translations can be implemented and momenta are not defined. Hence, in order to write a proper Hamiltonian, one is forced

to introduce a lattice in the configuration space and to approximate the momenta by the action of the translation operator on the minimum lattice distance (polymer scale). In this sense, polymer quantization naturally accounts for the presence of a fundamental discrete structure, as expected in quantum gravity approaches. As a consequence, the quantum dynamics is significantly affected at scales which are comparable with the polymer scale, while in the continuum limit there is a substantial overlap with the results obtained in the Schrödinger representation.

Let us address polymer quantization for ρ . The associated wave function in the frozen case can be written as

$$\Psi^{pol}(\rho, \phi) = \zeta^{pol}(\rho) e^{i \frac{k_\phi \phi}{\hbar}}, \quad (46)$$

where the superscript pol denotes the polymer part of the wave function. The function ζ^{pol} belongs to the Hilbert space of square-integrable functions over the Bohr compactification of the real line. It can be expanded on the eigenvectors of the polymer operator ρ , whose spectrum is defined on the lattice $\mathcal{L}_{\lambda_\star} = \{\rho = n\lambda_\star, n \in \mathbb{Z}\}$, *i.e.*

$$\hat{\rho} |n\lambda_\star\rangle = n\lambda_\star |n\lambda_\star\rangle, \quad (47)$$

λ_\star being the polymer scale. Finite translations act as

$$T_\rho |\rho'\rangle = |\rho' + \rho\rangle, \quad (48)$$

and the momentum operator can be defined in terms of them as follows

$$\hat{p}_\rho^{pol} = \frac{\hbar}{2i\lambda_\star} (T_{\lambda_\star} - T_{-\lambda_\star}), \quad (49)$$

which formally coincides with the replacement

$$\hat{p}_\rho \rightarrow \frac{1}{\lambda_\star} \sin\left(\frac{\lambda_\star \hat{p}_\rho}{\hbar}\right). \quad (50)$$

The eigenvalue problem (34) becomes

$$\left(\hat{H}^{PHO} + k_\phi^2 \frac{1}{\rho^2}\right) \zeta^{pol}(\rho) = -E \zeta^{pol}(\rho), \quad (51)$$

where \hat{H}^{PHO} is the Hamiltonian operator for the harmonic oscillator in the polymer representation, which in the momentum polarization reads [38]

$$\hat{H}^{PHO} \tilde{\zeta}^{pol}(p_\rho) = \frac{\hbar^2}{8M\lambda_\star^2} \left[2 - 2\cos\left(\frac{2\lambda_\star p_\rho}{\hbar}\right)\right] \tilde{\zeta}^{pol}(p_\rho) - \frac{\hbar^2}{2} M\Omega^2 \frac{d^2 \tilde{\zeta}^{pol}(p_\rho)}{dp_\rho^2}, \quad (52)$$

where M and Ω denote the mass and the frequency of the harmonic oscillator and they take the following expression in terms of the parameters of the model

$$M = \frac{1}{3\pi G}, \quad \Omega = \frac{\sqrt{3\Lambda}}{2}. \quad (53)$$

Let us assume that the second term on the left-hand side of (51) is a small perturbation and let us solve first the eigenvalue problem for the polymer harmonic oscillator:

$$\frac{\hbar^2}{8M\lambda_\star} \left[2 - 2\cos\left(\frac{2\lambda_\star p_\rho}{\hbar}\right)\right] \tilde{\zeta}^{pol} - \frac{\hbar^2}{2} M\Omega^2 \frac{d^2 \tilde{\zeta}^{pol}}{dp_\rho^2} = -E \tilde{\zeta}^{pol}. \quad (54)$$

By introducing the following quantities

$$u = \frac{\lambda_\star p_\rho}{\hbar} + \frac{\pi}{2}, \quad \alpha = -\frac{2E}{\hbar\Omega g} - \frac{1}{2g^2}, \quad g = \frac{M\Omega\lambda_\star^2}{\hbar}, \quad (55)$$

Eq. (54) takes the form of Mathieu equation

$$\frac{d^2 \tilde{\zeta}^{pol}}{du^2} + \left[\alpha - \frac{1}{2} g^{-2} \cos(2u)\right] \tilde{\zeta}^{pol} = 0. \quad (56)$$

We restrict to those solutions which are periodic (or antiperiodic) in u , since the associated conjugate variable ρ is discrete. These solutions of the Mathieu equation are parametrized by g and they can be written as

$$\tilde{\zeta}_{2n}^{pol}(u) = \pi^{-1/2} \text{ce}_n(u, g), \quad \alpha = A_n(g) \quad (57)$$

$$\tilde{\zeta}_{2n+1}^{pol}(u) = \pi^{-1/2} \text{se}_n(u, g), \quad \alpha = B_n(g) \quad (58)$$

where ce_n and se_n ($n \in \mathbb{N}$) are respectively the sine and cosine elliptic functions, while A_n and B_n are the *characteristic values functions*. For even n , ce_n and se_n are π -periodic, while for odd n they are π -antiperiodic. The energy eigenvalues are

$$E_{2n} = -\frac{\hbar\Omega g}{2} \left(A_n(g) + \frac{1}{2g^2} \right), \quad (59)$$

$$E_{2n+1} = -\frac{\hbar\Omega g}{2} \left(B_n(g) + \frac{1}{2g^2} \right). \quad (60)$$

It is worth noting how in our case the condition $g \ll 1$ holds, since we have

$$g = \lambda_\star^2 \frac{\sqrt{3\Lambda}}{3\pi\ell_P^2} \ll 1 \rightarrow \Lambda \ll \frac{3\pi^2}{\ell_P^2}, \quad (61)$$

where we fixed the polymer scale $\lambda_\star \sim \ell_P^{3/2}$. The condition above is an intrinsic consistency condition for our model, since if it was violated then the negative cosmological constant would drive the Universe evolution from the Planck time up to now. Hence, we can expand the characteristic values in powers of g near $g = 0$, so getting

$$\alpha = (2n + \frac{1}{2})g^{-1} - \frac{2n^2 + 2n + 1}{4} + O(g), \quad (62)$$

such that the energy spectrum becomes

$$E_n = -\frac{\hbar\sqrt{3\Lambda}}{2} \left(\left(n + \frac{1}{2} \right) - \frac{2n^2 + 2n + 1}{8}g + O(g^2) \right). \quad (63)$$

It is worth noting how the spectrum exhibits a minimum corresponding to

$$n_{min} \sim \frac{2}{g} \rightarrow E_{min} \sim \frac{\hbar\sqrt{3\Lambda}}{2g} = \frac{3\pi\hbar\ell_P^2}{2\lambda_\star^2}. \quad (64)$$

Therefore, the dynamical system is now endowed with a ground state, corresponding to the minimum energy eigenvalue, and one can assume the Universe to dynamically relax into such a state. Furthermore, the minimum eigenvalue is negative and the dual dust field energy density is positive; thus it behaves as an ordinary dust field contribution.

Let us now estimate the additional term in (51). By performing the Fourier transform of Mathieu functions it is possible to describe the eigenfunctions (57) and (58) in the coordinate representation. We consider only the Mathieu sine periodic functions, since the same estimate can be repeated for the cosine, and with odd index, since nothing changes for an even one. Hence, se_{2n+1} can be expanded as follows

$$\text{se}_{2n+1}(u, g) = \sum_{m=0}^{\infty} B_{2m+1}^{2n+1}(g) \sin[(2m+1)u], \quad (65)$$

where $B_{2m+1}^{2n+1}(g)$ are Fourier coefficients for which

$$\sum_{m=0}^{\infty} |B_{2m+1}^{2n+1}|^2 = 1. \quad (66)$$

The normalized energy eigenstates $|E_{2n+1}\rangle$, for which $\langle p_\rho | E_{2n+1} \rangle = \tilde{\zeta}_{2n+1}^{pol}$, can thus be expanded in terms of $|\rho\rangle = e^{i\rho p_\rho/\hbar} |p_\rho\rangle$ as follows:

$$|E_{2n+1}\rangle = \frac{1}{\sqrt{2}} \sum_{m=-\infty}^{+\infty} (-)^m B_{|2m+1|}^{2n+1} |(2m+1)\lambda_\star\rangle. \quad (67)$$

Inverse powers of ρ can be regularized in polymer representation using the expression [39]

$$\frac{\widehat{\text{sgn}(\rho)}}{\sqrt{|\rho|}} \rightarrow \frac{1}{\lambda_\star} \left(\sqrt{|\rho + \lambda_\star|} - \sqrt{|\rho - \lambda_\star|} \right), \quad (68)$$

which provides for $1/\rho^2$

$$\frac{\widehat{1}}{\rho^2} \rightarrow \frac{1}{\lambda_\star^4} \left(\sqrt{|\rho + \lambda_\star|} - \sqrt{|\rho - \lambda_\star|} \right)^4. \quad (69)$$

Hence, the expectation value of the additional term in (51) on energy eigenstates reads

$$k_\phi^2 \left\langle E_{2n+1} \left| \frac{\widehat{1}}{\rho^2} \right| E_{2n+1} \right\rangle = \frac{k_\phi^2}{\lambda_\star^2} \sum_{m=0}^{\infty} |B_{2m+1}^{2n+1}|^2 \left(\sqrt{|2m+2|} - \sqrt{|2m|} \right)^4 \leq \frac{k_\phi^2}{\lambda_\star^2} \sum_{m=0}^{\infty} \left(\sqrt{|2m+2|} - \sqrt{|2m|} \right)^4, \quad (70)$$

where we used the condition (66). Let us note that

$$\left(\sqrt{n+1} - \sqrt{n-1} \right)^4 < \frac{2}{n^2} \quad \forall n > 1, \quad (71)$$

and thus

$$\sum_{m=0}^{\infty} \left(\sqrt{|2m+2|} - \sqrt{|2m|} \right)^4 < 4 + \sum_{m=1}^{\infty} \frac{2}{(2m+1)^2} = 2 + \sum_{m=0}^{\infty} \frac{2}{(2m+1)^2} = 2 + \frac{\pi^2}{4}, \quad (72)$$

such that for the expression (70) one has

$$k_\phi^2 \left\langle E_{2n+1} \left| \frac{\widehat{1}}{\rho^2} \right| E_{2n+1} \right\rangle < \frac{8 + \pi^2}{4} \frac{k_\phi^2}{\lambda_\star^2}. \quad (73)$$

The ratio Δ of this term with the ground state energy eigenvalue reads

$$\Delta = \frac{k_\phi^2 \left\langle E_{2n+1} \left| \frac{\widehat{1}}{\rho^2} \right| E_{2n+1} \right\rangle}{E_{min}} < \frac{8 + \pi^2}{4} k_{\phi\star}^2, \quad (74)$$

and it tells us if the additional term is actually a small perturbation or not. For instance, we can require $\Delta < 10^{-1}$, so getting

$$k_{\phi\star} < 0.15. \quad (75)$$

We see how the smaller the value of $k_{\phi\star}$, the more accurate the energy spectrum (63) we got in polymer representation.

VI. CONCLUSION

We presented a self-consistent cosmological picture, based on the implementation of an evolutionary quantum gravity approach to a reliable model of the primordial isotropic Universe. Indeed, we construct a nonsingular cosmology, corresponding to a cyclic Universe, having a quantum big bounce in the past, associated to the details of its quantum dynamics, and a late turning point, due to the presence of a small negative cosmological constant. The main issue of the analysis above consists in determining, via a polymer quantum approach, a ground state of the Universe, associated to a positive dust energy density and to a high occupation number. For such a vacuum state of the model, it is possible to construct a reliable classical limit, which reconciles the early evolutionary Universe phase with a standard preinflationary scenario (with a decay from a false vacuum). Furthermore, the emerging dust energy density is strongly redshifted by the Universe de Sitter phase and thus, the postinflationary dynamics is indistinguishable from the Standard Model one. Finally, we observe that the value of the negative cosmological constant does not enter the ground state energy of the Universe and so it can be easily stated as an amount which is unable to affect the present Universe dynamics. We can consequently conclude that an evolutionary quantum cosmology exists, able to solve the singular nature of the big

bang, without any dynamical discrepancy with respect to a standard Friedmann Universe and with the additional feature of a natural picture of the model classical limit.

Despite the fact that the present study offers an interesting cosmological paradigm for the matter-time dualism and the related evolutionary quantum gravity, we stress how the problem of a unique definition of a physical clock for the dynamics of the gravitational field in canonical quantum gravity remains one of the most challenging topics in this area and deserves attention for both its fundamental developments and specific applications.

We conclude by observing how the relevance of the present model must be also recognized in its possibly very general character. In fact, the anisotropic Universe degrees of freedom are dynamically equivalent to a scalar field and the associated potential (due to the cosmological model spatial curvature) is quadratic in the limit of a quasi-isotropic Universe. Then, we could first extend the present model to the homogeneous Bianchi Universe (in particular to the Bianchi IX model, generalizing the closed isotropic Universe) and then to a generic inhomogeneous Universe, via a quantum version of the so-called Belinski-Lipschitz-Khalatnikov conjecture [40, 41].

Although, differently from the scalar field dynamics, there is no firm evidence that the limit of small anisotropies is prescribed by the Universe evolution, the perspective to replace the scalar field with the cosmological gravitational field degrees of freedom, opens very general and intriguing scenarios for the implementation of the idea traced here.

VII. APPENDIX: BORN-OPPENHEIMER APPROXIMATION

We demonstrate that the solutions of the evolutionary equation (11) in the large ρ limit can be constructed as the product of the two harmonic oscillator solutions (14) and (18). In particular, by inserting the expression (21) into Eq. (11) one gets

$$\begin{aligned} & \left[\frac{3\pi\ell_P^4}{2G} \frac{\partial^2}{\partial\rho^2} - \frac{\Lambda}{8\pi G} \rho^2 - \frac{\hbar}{2\rho^2} \frac{\partial^2}{\partial\phi^2} + \frac{1}{2} \frac{m^2}{\hbar} \rho^2 \phi^2 \right] \zeta_n(\rho) \chi_k(\rho, \phi) \\ &= \frac{3\pi\ell_P^4}{2G} \{ \zeta_n''(\rho) \chi_k(\rho, \phi) + \zeta_n(\rho) \chi_k''(\rho, \phi) + 2\zeta_n'(\rho) \chi_k'(\rho, \phi) \} \\ & - \frac{\hbar}{2\rho^2} \zeta_n(\rho) \frac{\partial^2}{\partial\phi^2} \chi_k(\rho, \phi) + \left(\frac{1}{2} \frac{m^2}{\hbar} \rho^2 \phi^2 - \frac{3\pi\ell_P^4}{2G} \Lambda_\star \rho^2 \right) \zeta_n(\rho) \chi_k(\rho, \phi) \\ &= E \zeta_n(\rho) \chi_k(\rho, \phi), \end{aligned} \quad (76)$$

where $'$ denotes the derivative with respect to the variable ρ . These derivatives can be evaluated thanks to the following identity for Hermite polynomials

$$\frac{\partial}{\partial\rho} H_n(\rho) = 2n H_{n-1}(\rho), \quad (77)$$

and the following asymptotic recurrence relation holding as soon as $\rho \rightarrow \infty$

$$H_{n-1}(\rho) \simeq \frac{1}{2\rho} H_n(\rho), \quad (78)$$

which can be combined together, so getting

$$\frac{\partial}{\partial\rho} H_n(\rho) \simeq \frac{n}{\rho} H_n(\rho). \quad (79)$$

Hence, $\zeta_n'(\rho)$ in the limit $\rho \rightarrow \infty$ reads

$$\zeta_n'(\rho) \simeq \left[-\sqrt{\Lambda_\star} \rho + \frac{n}{\rho} \right] \zeta_n(\rho), \quad (80)$$

and by iterating $\zeta_n''(\rho)$ can be estimated, finding

$$\zeta_n''(\rho) \simeq \left[\Lambda_\star \rho^2 - 2\sqrt{\Lambda_\star} \left(n + \frac{1}{2} \right) + \frac{n(n-1)}{\rho^2} \right] \zeta_n(\rho). \quad (81)$$

The same procedure gives the following expressions for $\chi_k'(\rho, \phi)$ and $\chi_k''(\rho, \phi)$

$$\chi_k'(\rho, \phi) \simeq \left[-m_\star \phi^2 \rho + \frac{k + \frac{1}{2}}{\rho} \right] \chi_k(\rho, \phi), \quad (82)$$

$$\chi_k''(\rho, \phi) \simeq \left\{ m_\star^2 \phi^4 \rho^2 - 2m_\star (k+1) \phi^2 + \frac{(k+\frac{1}{2})(k-\frac{1}{2})}{\rho^2} \right\} \chi_k(\rho, \phi). \quad (83)$$

By using the relations above, the eigenvalue problem (76) becomes

$$\frac{3\pi\ell_P^4}{2G} \zeta_n''(\rho) \chi_k(\rho, \phi) - \frac{\hbar}{2\rho^2} \zeta_n(\rho) \frac{\partial^2}{\partial\phi^2} \chi_k(\rho, \phi) + \Upsilon(\rho, \phi) \zeta_n(\rho) \chi_k(\rho, \phi) = E \zeta_n(\rho) \chi_k(\rho, \phi). \quad (84)$$

where

$$\begin{aligned} \Upsilon(\rho, \phi) = & \underbrace{\frac{\sqrt{3}}{2} \hbar \sqrt{\Lambda} \left(k + \frac{1}{2}\right)}_a - \underbrace{\frac{\Lambda}{8\pi G} \rho^2}_b + \underbrace{\frac{3\pi\ell_P^4}{2G} \frac{1}{\rho^2} \left(k + \frac{1}{2}\right) \left[2n + k - \frac{1}{2}\right]}_c \\ & + \underbrace{\left(\frac{1}{2} \frac{m^2}{\hbar} + \frac{\sqrt{3}}{2} m \sqrt{\Lambda}\right) \rho^2 \phi^2}_d - \underbrace{3\pi\ell_P^2 m (n+k+1) \phi^2}_e + \underbrace{\frac{3\pi\ell_P^2}{2\hbar} m^2 \rho^2 \phi^4}_f. \end{aligned} \quad (85)$$

We proceed to the analysis of each single term in (85):

- a) The constant terms redefine the energy eigenvalue;
- b) The terms of ρ^2 order is the leading one as $\rho \rightarrow \infty$;
- c) The terms proportional to $\frac{1}{\rho^2}$ are negligible as soon as

$$\left| E - \frac{\sqrt{3}}{2} \hbar \sqrt{\Lambda} \left(k + \frac{1}{2}\right) \right| \gg \frac{3\pi\ell_P^4}{2G} \frac{1}{\rho^2} \left(k + \frac{1}{2}\right) \left(2n + k - \frac{1}{2}\right). \quad (86)$$

- d) The terms $\rho^2 \phi^2$ determine the quadratic part of the effective scalar field potential and, to preserve the solution (18), we must impose

$$m \gg \hbar \sqrt{3\Lambda}, \quad (87)$$

which tell us that the square root of the cosmological constant must be negligible with respect to the mass of the inflaton in the false vacuum.

- e) The contribution of order $\rho^0 \phi^2$ can be neglected with respect to that of order $\rho^2 \phi^2$ as soon as

$$\frac{1}{2} m^2 \rho^2 \gg 3\pi\ell_P^2 m (n+k+1). \quad (88)$$

- f) The term proportional to $\rho^2 \phi^4$ can be seen as a quartic potential contribution for the scalar field and it can be neglected if

$$\phi^2 \ll \frac{1}{3\pi\ell_P^2}, \quad (89)$$

which means that the scalar field takes sub-Planckian values (this is basically the reason why we cannot apply this framework to chaotic inflation [42]).

Therefore, as soon as ρ is sufficiently big that the conditions (86) and (88) hold, Eq. (76) reduces to

$$\begin{aligned} & \frac{3\pi\ell_P^4}{2G} \zeta_n''(\rho) \chi_k(\rho, \phi) - \frac{\hbar}{2\rho^2} \zeta_n(\rho) \frac{\partial^2}{\partial\phi^2} \chi_k(\rho, \phi) + \\ & \left(-\frac{\Lambda}{8\pi G} \rho^2 + \frac{1}{2} m^2 \rho^2 \phi^2 + \frac{\sqrt{3}}{2} \hbar \sqrt{\Lambda} \left(k + \frac{1}{2}\right) \right) \zeta_n(\rho) \chi_k(\rho, \phi) = E \zeta_n(\rho) \chi_k(\rho, \phi). \end{aligned} \quad (90)$$

from which it follows that (21) is the proper eigenfunction with eigenvalue (22).

If the Universe energy eigenvalue is preserved, we can take for E the value (64) (for $\lambda_\star = \ell_P^{3/2}$) and estimate n via the relation (22) (we assume the contribution of the scalar field to be negligible). Hence, from (86) we get

$$\rho^2 \gg 2n \ell_P^3 \rightarrow a \gg 10^{-6} m, \quad (91)$$

which fixes a lower bound for the age in which the decay into the true vacuum (thus also inflation) starts.

Acknowledgments

The work of F. C. was supported by funds provided by the National Science Center under the agreement DEC12 2011/02/A/ST2/00294.

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- [1] C. Rovelli, *Class. Quant. Grav.* **8**, 297 (1991).
 - [2] C. J. Isham, In **Salamanca 1992, Proceedings, Integrable systems, quantum groups, and quantum field theories** 157-287, and *London Imp. Coll. - ICTP-91-92-25 (92/08,rec.Nov.)* 124 p [gr-qc/9210011].
 - [3] F. Cianfrani, O. M. Lecian, M. Lulli, G. Montani, “Fundamental and recent developments in canonical Quantum Gravity,” (World Scientific, 2014).
 - [4] C. Rovelli, *Class. Quant. Grav.* **8**, 317 (1991).
 - [5] K. V. Kuchar and C. G. Torre, *Phys. Rev. D* **43**, 419 (1991).
 - [6] J. D. Brown and K. V. Kuchar, *Phys. Rev. D* **51**, 5600 (1995) [gr-qc/9409001].
 - [7] S. Mercuri and G. Montani, *Mod. Phys. Lett. A* **19**, 1519 (2004) [gr-qc/0312077].
 - [8] V. A. Savchenko, T. P. Shestakova and G. M. Vereshkov, *Grav. Cosmol.* **7**, 18 (2001) [gr-qc/9809086].
 - [9] G. Montani, *Nucl. Phys. B* **634**, 370 (2002) [gr-qc/0205032].
 - [10] G. Montani and F. Cianfrani, *Class. Quant. Grav.* **25**, 065007 (2008) [arXiv:0802.0942 [gr-qc]].
 - [11] G. Montani, N. Carlevaro, F. Cianfrani and V. Lacquaniti, arXiv:0904.0568 [gr-qc].
 - [12] G. Corvino and G. Montani, *Mod. Phys. Lett. A* **19**, 2777 (2004) [gr-qc/0410026].
 - [13] M. V. Battisti and G. Montani, *Phys. Lett. B* **637**, 203 (2006) [gr-qc/0604049].
 - [14] E. W. Kolb and M. S. Turner, “The Early Universe,” (Westview Press, 1994).
 - [15] G. Montani, M. V. Battisti, R. Benini and G. Imponente, “Primordial cosmology,” Hackensack, USA: World Scientific (2011) 591 p
 - [16] R. Bousso, D. Harlow and L. Senatore, arXiv:1309.4060 [hep-th].
 - [17] R. Bousso, D. Harlow and L. Senatore, arXiv:1404.2278 [astro-ph.CO].
 - [18] D. Baumann, arXiv:0907.5424 [hep-th].
 - [19] A. Ashtekar, S. Fairhurst and J. L. Willis, *Class. Quant. Grav.* **20**, 1031 (2003) [gr-qc/0207106].
 - [20] A. Corichi, T. Vukasinac and J. A. Zapata, *Phys. Rev. D* **76**, 044016 (2007) [arXiv:0704.0007 [gr-qc]].
 - [21] M. V. Battisti, O. M. Lecian and G. Montani, *Phys. Rev. D* **78**, 103514 (2008) [arXiv:0806.0768 [gr-qc]].
 - [22] C. Rovelli, Cambridge, UK: Univ. Pr. (2004) 455 p
 - [23] T. Thiemann, Cambridge, UK: Cambridge Univ. Pr. (2007) 819 p [gr-qc/0110034].
 - [24] C. Rovelli, *Phys. Rev. D* **43**, 442 (1991).
 - [25] T. Thiemann, astro-ph/0607380.
 - [26] F. Cianfrani, G. Montani and S. Zonetti, *Class. Quant. Grav.* **26**, 125002 (2009) [arXiv:0807.3281 [gr-qc]].
 - [27] C. J. Isham and K. V. Kuchar, *Annals Phys.* **164**, 316 (1985).
 - [28] K. Kuchar, “Canonical Methods of Quantization,” In Oxford 1980, *Proceedings, Quantum Gravity 2*, 329-376.
 - [29] A. G. Riess *et al.* [Supernova Search Team Collaboration], *Astron. J.* **116**, 1009 (1998) [astro-ph/9805201].
 - [30] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], *Astrophys. J.* **517**, 565 (1999) [astro-ph/9812133].
 - [31] G. Hinshaw *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **208**, 19 (2013) [arXiv:1212.5226 [astro-ph.CO]].
 - [32] S. Tsujikawa, *Class. Quant. Grav.* **30**, 214003 (2013) [arXiv:1304.1961 [gr-qc]].
 - [33] B. de Wit and I. Herger, *Lect. Notes Phys.* **541**, 79 (2000) [hep-th/9908005].
 - [34] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, *Lett. Math. Phys.* **99**, 3 (2012) [arXiv:1012.3982 [hep-th]].
 - [35] Zettili, N., “Quantum Mechanics Concepts and Applications”, 2009 John Wiley & Sons, Ltd
 - [36] F. Calogero, *J. Math. Phys.* **10**, 2191 (1969).
 - [37] L.D. Landau and E. Lifshitz, “Theoretical Physics 3, Quantum Mechanics”, (Addison-Wesley, New York, 1975).
 - [38] G. M. Hossain, V. Husain and S. S. Seahra, *Phys. Rev. D* **82**, 124032 (2010) [arXiv:1007.5500 [gr-qc]].
 - [39] V. Husain, J. Louko and O. Winkler, *Phys. Rev. D* **76**, 084002 (2007) [arXiv:0707.0273 [gr-qc]].
 - [40] V. a. Belinsky, I. m. Khalatnikov and E. m. Lifshitz, *Adv. Phys.* **31**, 639 (1982).
 - [41] A. A. Kirillov and G. Montani, *JETP Lett.* **66**, 475 (1997).
 - [42] A. D. Linde, *Phys. Lett. B* **129**, 177 (1983).
 - [43] The units are $[\phi] = (\text{length}^{-1})$ and $[m] = \text{mass}$.
 - [44] The scale factor a - and consequently $\rho = a^{3/2}$ - cannot take negative values while harmonic oscillator solutions (14) do admit them. We must then impose that n takes only odd values in \mathbb{N} , corresponding to a wave function node (21) in $\rho = 0$, and consider only the values $\rho \geq 0$.
 - [45] The two functions $e^{-\frac{\sqrt{\Lambda_*}\rho^2}{2}}$ and ρ^λ entering (36) are the solutions of the associated Wheeler-DeWitt equation in the two limits $\rho \rightarrow \infty$ and $\rho \rightarrow 0$.
 - [46] In quantum mechanics the singularity takes the significance of finding the particle exactly localized in the source, phenomenon also known as “fall to the center”.